POINCARÉ SERIES OF ALGEBRAIC LINKS AND LATTICE HOMOLOGY

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ABSTRACT. We construct a version of the lattice homology for plane curve singularities using the normalization of their components. We prove that the Poincaré series of the associated graded homologies can be identified by an algebraic procedure with the motivic Poincaré series. Hence, for a plane curve singularity the following objects carry the same information: the multi-variable Alexander polynomial, the multi-variable Hilbert series associated with the normalization, the motivic Poincaré series, and the Poincaré series of the newly introduced lattice homology. We also conjecture a relation of this lattice homology with the Heegaard–Floer homology of the corresponding link.

1. INTRODUCTION

Let $C=\cup_{i=1}^{r}C_i$ be a reduced plane curve singularity at the origin in $\mathbb{C}^2$, where C_i are the irreducible components. Let $\gamma_i:(\mathbb{C},0)\to(C_i,0)$ be some uniformizations of these components. We define r integer-valued functions on the $\mathbb{C}$–algebra $\mathcal{O}=\mathcal{O}_{\mathbb{C}^2,0}$ by

$$\mathfrak{v}_i(f)=\text{Ord}\left(f\left(\gamma_i(t)\right)\right),$$

and a $\mathbb{Z}^r$-indexed filtration

$$J(v)=\{f\in\mathcal{O}|\mathfrak{v}_i(f)\geq v_i \text{ for all } i\}.$$

Campillo, Delgado and Gusein-Zade considered in [4] the Poincaré series of the filtration $J(v)$ defined as the integral with respect to the Euler characteristic over the projectivization of $\mathcal{O}$:

$$P(t_1,\dots,t_r)=\int_{\mathbb{P}\mathcal{O}}t_1^{\mathfrak{v}_1}\cdot\ldots\cdot t_r^{\mathfrak{v}_r}d\chi. \tag{1.0.1}$$

The Poincaré series P can be related with several topological and analytical objects.

For example, Campillo, Delgado and Gusein-Zade have shown in [4] that P basically is the multi-variable Alexander polynomial of the link of C (the intersection of C with a small three-dimensional sphere centered at the origin). For the precise statement see Theorem 2.1.1.

A more analytic invariant is the Hilbert series H associated with the filtration $\{J(v)\}_v$, it has coefficients $h(v)=\dim\mathcal{O}/J(v)$, see Definition 2.2.1. It is known that P and H determine each other, see [3, 11], in section 2 we reprove this fact.

Finally, we consider the motivic version of the Poincaré series as well ([5, 7, 12]):

$$P_g(t_1,\dots,t_r;q)=\int_{\mathbb{P}\mathcal{O}}t_1^{\mathfrak{v}_1}\cdot\ldots\cdot t_r^{\mathfrak{v}_r}d\mu. \tag{1.0.2}$$

Here μ denotes the motivic measure on $\mathbb{P}\mathcal{O}$ [5, 7]. Since P_g can be derived from the coefficients of H as well, see subsection 2.4, P_g carries the same information as H and P. Nevertheless, all these invariants capture different aspects and highlight different geometrical structures of the local plane curve singularities.

This note introduces another object, the lattice cohomology of the singularity. In [17] the lattice cohomology of a normal surface singularity was introduced via the lattice provided by its resolution graph (or plumbing graph of the link). The invariant created a bridge between the

analytic invariants of the singularity with several topological ones of the link, namely Seiberg-Witten and Heegaard-Floer theories. The goal of the present construction is similar; nevertheless here we rely on a different lattice: the needed weight function is provided by the normalization, by the coefficients $h(v)$. (In order to eliminate any further confusion, we will call the present invariant "lattice homology via normalization").

In short, the definition runs as follows. The lattice complex $\mathcal{L}^-$ is generated over $\mathbb{Z}[U]$ by elementary cubes $\square$ of all dimensions in $\mathbb{R}^r$, with vertices in the lattice $\mathbb{Z}^r$. For such a cube we define $h(\square) = \max_{x \in \square \cap \mathbb{Z}^r} h(x)$. The differential is defined as

$$\partial_U(\square) = \sum_i \varepsilon_i U^{h(\square)-h(\square_i)} \square_i,$$

where $\square_i$ are the oriented boundary cubes of $\square$, and ε_i are the corresponding signs.

The complex $\mathcal{L}^-$ is naturally $\mathbb{Z}^r$-filtered: the subcomplex $\mathcal{L}^-(v)$ is generated by the cubes contained in the positive quadrant originating at v. Our main theorem describes the homology of the subcomplexes $\mathcal{L}^-(v)$ and the associated graded complexes $\operatorname{gr}_v \mathcal{L}^-$ for all v.

Theorem 1.0.3. (1) *The homology of $\mathcal{L}^-(v)$ is given by $H^*(\mathcal{L}^-(v)) = \mathbb{Z}[U][-2h(v)]$. Hence, the Poincaré series for this homology equals*

$$P_{\mathcal{L}^-(v)}(t) = \frac{t^{-2h(v)}}{1-t^{-2}}.$$

(2) *The Poincaré polynomial of the homology of $\operatorname{gr}_v \mathcal{L}^-$ is given by the formula*

$$P_{\operatorname{gr}_v \mathcal{L}^-}(t) = (-t)^{-h(v)} H_v(-t^{-1}),$$

where $H_v(q)$ is the coefficient in the motivic Poincaré series:

$$P_g(t_1, \ldots, t_r; q) = \sum_v H_v(q) t_1^{v_1} \cdots {t_r}^{v_r}.$$

We prove the parts of this theorem as Theorem 4.2.2 and Theorem 4.3.2. The main ingredient is the structure of the Orlik-Solomon algebras associated with central hyperplane arrangements.

Corollary 1.0.4. *The polynomial $(-1)^{h(v)} H_v(-t)$ has only non-negative coefficients.*

Corollary 1.0.5. *The generating function for the Euler characteristics of $\operatorname{gr}_v \mathcal{L}^-$,*

$$\sum_v P_{\operatorname{gr}_v \mathcal{L}^-}(-1) \cdot t_1^{v_1} \cdots t_r^{v_r},$$

equals $P(t_1, \ldots, t_r)$. Hence, it can be expressed by the multi-variable Alexander polynomial.

Motivated by Theorem 1.0.3 and Corollary 1.0.5, we formulate the following

Conjecture 1.0.6. *The graded homology of $\operatorname{gr} \mathcal{L}^-$ is isomorphic to the Heegaard-Floer homology HF^- of the link of C (*[21, 22, 23, 24]*, see also* [26]*).*

We compare various structural properties of both homology theories and show that the conjecture holds for $r = 1$. We also compute the homology of $\operatorname{gr} \mathcal{L}^-$ in other examples and match them to the Heegaard-Floer homology.

ACKNOWLEDGEMENTS

The authors are grateful to S. Gusein-Zade, J. Moyano-Fernandez, P. Ozsváth and J. Rasmussen for the useful discussions. We also thank the Oberwolfach Mathematical Institute, where part of this work was done, for the hospitality. The research of E. G. is partially supported by the grants RFBR-10-01-678, NSh-8462.2010.1 and the Simons foundation. A. N. is partially supported by OTKA Grants 81203 and 100796.

2. HILBERT FUNCTION

2.1. **The Poincaré series of multi-index filtrations.** Let us fix a local plane curve singularity with r irreducible components as in the introduction. Set $K_0 = \{1, \dots, r\}$. Let e_i denote the i-th coordinate vector in $\mathbb{Z}^r$. For a subset $K \subset K_0$ we write $e_K = \sum_{i \in K} e_i$ and $e = e_{K_0} = \sum e_i$. For a vector v we set $v_K = \sum_{i \in K} v_i e_i$.

Furthermore, let P be the series defined in (1.0.1), and $\Delta(t_1, \dots, t_r)$ be the multi-variable Alexander polynomial of the link of C. Then the following holds.

Theorem 2.1.1 ([4]). *If $r = 1$, then $P(t)(1-t) = \Delta(t)$, while if $r > 1$, then*

$$P(t_1, \dots, t_r) = \Delta(t_1, \dots, t_r). \tag{2.1.2}$$

2.2. **The Hilbert series of multi-index filtrations.** We set a partial order on $\mathbb{Z}^r$ by

$$u \preceq v \iff u_i \leq v_i \quad \text{for all } i.$$

We define r integer-valued functions on $\mathcal{O} = \mathcal{O}_{\mathbb{C}^2,0}$ by $\mathfrak{v}_i(f) = \mathrm{Ord}_0\left(f\left(\gamma_i(t)\right)\right)$, and a $\mathbb{Z}^r$-indexed filtration

$$J(v) = \{f \in \mathcal{O} \mid \mathfrak{v}(f) \succeq v\}.$$

Note that the ideals $J(v)$ are also defined for negative values of v. This filtration is decreasing in the sense that if $u \preceq v$, then $J(u) \supset J(v)$.

Definition 2.2.1. Let $h(v) = \dim \mathcal{O}/J(v)$. We define the Hilbert series of a multi-index filtration J as the series

$$H(t_1, \dots, t_r) = \sum_v h(v) \cdot t_1^{v_1} \cdots t_r^{v_r}. \tag{2.2.2}$$

Furthermore, we define the set $\mathcal{S} := \{v \in \mathbb{Z}^r \ : \ \text{there exists } f \in \mathcal{O} \text{ with } \mathfrak{v}(f) = v\}$ as well. It is called the *semigroup of f*. From the definition one gets the following elementary facts.

Lemma 2.2.3. *$\mathcal{S} = \{v \in \mathbb{Z}^r_{\geq 0} \ : \ h(v + e_i) > h(v)$ for every $i = 1, \dots, r\}$. Furthermore, fix any $0 \preceq v$ and e_i. Then $h(v + e_i) = h(v) + 1$ if there is an element $u \in \mathcal{S}$ such that $u_i = v_i$ and $u_j \geq v_j$ for $j \neq i$. Otherwise $h(v + e_i) = h(v)$. In particular, H and $\mathcal{S}$ determine each other.*

Proof. If $h(v + e_i) > h(v)$ for all i, then there exist functions f_i such that $\mathfrak{v}_i(f_i) = v_i$ and $\mathfrak{v}_j(f_i) \geq v_j$ for $j \neq i$. Therefore $\mathfrak{v}(\sum_{i=1}^r \lambda_i f_i) = v$ for generic coefficients λ_i. For the second part note that $h(v+e_i) - h(v) = \dim J(v)/J(v+e_i)$. This quotient space is trivial if there is no function f such that $\mathfrak{v}_i(f) = v_i$ and $\mathfrak{v}_j(f) \geq v_j$ for $j \neq i$. Otherwise it is one-dimensional. □

The following lemma is a variation of the analogous statement from [3].

Lemma 2.2.4. *The series H and P are related by*

$$P(t_1, \dots, t_r) = -H(t_1, \dots, t_s) \cdot \prod_i (1 - t_i^{-1}). \tag{2.2.5}$$

At the level of coefficients this reads as follows. Define the integers $\pi(v)$ by the equation

$$P(t_1,\dots,t_r)=\sum_v t_1^{v_1}\dots t_r^{v_r}\cdot\pi(v). \tag{2.2.6}$$

Then

$$\pi(v)=\sum_{K\subset K_0}(-1)^{|K|-1}h(v+e_K). \tag{2.2.7}$$

We will invert the equation (2.2.7), namely, we express the integers $h(v)$ from the Poincaré series P_C. First note that

$$h(v)=h(\max(v,0)), \tag{2.2.8}$$

where $\max(v,0)_i=v_i$ if $v_i>0$ and $=0$ otherwise. Hence it is enough to determine $h(v)$ for $0\preceq v$ only. The series $\sum_{0\preceq v}h(v)t^v$ will be denoted by $H(t)|_{0\preceq v}$.

Next, for a subset $K=\{i_1,\dots,i_{|K|}\}\subset K_0$, $K\neq\emptyset$, consider a curve $C_K=\cup_{i\in K}C_i$. As above, this germ defines a $|K|$-index filtration of $\mathcal{O}_{\mathbb{C}^2,0}$, hence it provides the Hilbert series H_{C_K} of C_K in variables $\{t_i\}_{i\in K}$:

$$H_{C_K}(t_{i_1},\dots,t_{i_{|K|}})=\sum_v t_{i_1}^{v_{i_1}}\dots t_{i_{|K|}}^{v_{i_{|K|}}}\cdot h^K(v).$$

By construction, for $K\subset K_0$ one has $H_{C_K}(t_{i_1},\dots,t_{i_{|K|}})=H_C(t_1,\dots,t_r)|_{t_i=0\ i\notin K}$; or

$$\text{if } v_i=0 \text{ for all } i\notin K\text{, then } \ h^K(v)=h(v). \tag{2.2.9}$$

Analogously, we have the Poincaré series of C_K:

$$P^K(t_1,\dots,t_r)=P_{C_K}(t_{i_1},\dots,t_{i_{|K|}})=\sum_v t_{i_1}^{v_{i_1}}\dots t_{i_{|K|}}^{v_{i_{|K|}}}\cdot\pi^K(v).$$

By definition, for $K=\emptyset$ we take $\pi^{\emptyset}(v)=0$. Note that from $h(v)$ one can recover any P^K: first computing h^K by (2.2.9) and then recovering P^K from H^K using (2.2.7).

Although by [29] P determines the embedded topological type of C, hence all the series P^K for all subsets $K\subset K_0$ as well, the analogue of (2.2.9) is not true for the pair $\pi^K(v)$ and $\pi(v)$. Indeed, the "restricting relation" ([28]) is of type

$$P^{K_0\setminus\{1\}}(t_2,\dots,t_r)=P(t_1,\dots,t_r)|_{t_1=1}\cdot\frac{1}{(1-t_2^{(C_1,C_2)})\cdots(1-t_r^{(C_1,C_r)})}, \tag{2.2.10}$$

where (C_i,C_j) denotes the intersection multiplicity at the origin of the components C_i and C_j, $i\neq j$. This also shows that given these intersection multiplicities one can recover by induction P^K from P easily.

The following theorem was proved in [11] as Corollary 4.3. For the completeness of the exposition we present a proof of it (which is slightly different from [11]).

Theorem 2.2.11. *The following equation holds:*

$$H(t_1,\dots,t_r)|_{0\preceq v}=\frac{1}{\prod_{i=1}^r(1-t_i)}\sum_{K\subset K_0}(-1)^{|K|-1}\Big(\prod_{i\in K}t_i\Big)\cdot P^K(t_1,\dots,t_r). \tag{2.2.12}$$

Proof. The identity (2.2.12) for the generating series is equivalent to the following identity for their coefficients:

$$h(v)=\sum_{K\subset K_0}(-1)^{|K|-1}\sum_{0\preceq u\preceq v_K-e_K}\pi^K(u). \tag{2.2.13}$$

We will prove the identity (2.2.13) by a two-step induction: the first induction is by the number of components r, and the second one (for fixed r) is over the norm $|v| = \sum v_i$.

If $r = 1$, then (2.2.7) implies that $\pi(v) = h(v+1) - h(v)$. Therefore $\sum_{0\le u\le v-1} \pi(u) = h(v)$ since $h(0) = 0$.

Let us prove (2.2.13) for the case when at least one of coordinates v_i vanish. We can assume that $v_r = 0$. By the induction assumption we get

$$h(v) = h^{\{1,\dots,r-1\}}(v_1, \dots, v_{r-1}) = \sum_{K\subset\{1,\dots r-1\}} (-1)^{|K|-1} \sum_{0\preceq u\preceq v_K - e_K} \pi^K(u).$$

On the other hand, in (2.2.13) for all $K \subset K_0$ with $r \in K$ we get the vacuous restriction $0 \le u_r \le -1$, hence we get nontrivial contribution only from terms with $K \subset \{1, \dots r-1\}$.

Suppose now that v has no vanishing coordinates and we proved (2.2.13) for $v - e_K$ for all non-empty subsets $K \subset K_0$. We can rewrite (2.2.7) as a linear equation on $h(v)$:

$$\pi(v-e) = \sum_{K\subset K_0} (-1)^{r-|K|-1} h(v - e_K).$$

By the induction assumption for $K \neq \emptyset$ we have

$$h(v - e_K) = \sum_{M\subset K_0} (-1)^{|M|-1} \sum_{0\preceq u\preceq (v_M - e_{K\cap M} - e_M)} \pi^M(u),$$

and we should establish the same identity for $K = \emptyset$. Therefore we have to prove that

$$\pi(v-e) = \sum_{K\subset K_0} \sum_{M\subset K_0} (-1)^{r-|K|+|M|} \sum_{0\preceq u\preceq (v_M - e_{K\cap M} - e_M)} \pi^M(u). \tag{2.2.14}$$

Let us fix M and $u \preceq v - e$ and sum the expression $(-1)^{|K|}$ over all sets $K \subset K_0$ such that $u_i \le v_i - 2$ for $i \in K \cap M$. This sum does not vanish iff $M = K_0$ and $u_i = v_i - 1$ for all i. This proves (2.2.14). □

Corollary 2.2.15. *The restricted Hilbert function $H(t)|_{0\preceq v}$ of a multi-component curve is a rational function with denominator $\prod_{i=1}^r (1-t_i)^2$.*

Example 2.2.16. Consider the singularity A_{2n-1}. Its Poincaré series is $1 + t_1t_2 + \dots + (t_1t_2)^{n-1}$, and the Poincaré series of both its components equals $1/(1-t)$. The Hilbert series is

$$H(t_1,t_2)|_{0\preceq v} = \frac{1}{(1-t_1)(1-t_2)} \left(\frac{t_1}{1-t_1} + \frac{t_2}{1-t_2} - t_1t_2(1 + \ldots + (t_1t_2)^{n-1}) \right).$$

Therefore, for non-negative integers (v_1, v_2) one has

$$h(v) = \begin{cases} \max(v_1, v_2), & \text{if } \min(v_1,v_2) < n \\ v_1 + v_2 - n, & \text{otherwise.} \end{cases}$$

Figure 1 illustrates this formula for the Hilbert function for A_3 singularity. The points corresponding to the semigroup $\mathcal{S}$ are marked in bold.

Example 2.2.17. Consider the singularity D_5, that is, equation $y \cdot (x^2 - y^3) = 0$. Then

$$P(t_1,t_2) = 1 + t_1t_2^3, P_1(t_1) = \frac{1}{1-t_1}, P_2(t_2) = \frac{1 - t_2 + t_2^2}{1-t_2}.$$

4	4	**4**	**5**	**6**
3	3	**3**	**4**	**5**
2	2	**2**	**3**	**4**
1	**1**	2	3	4
0	1	2	3	4

FIGURE 1. Values of the Hilbert function for A_3

4	4	**4**	**5**	**6**	**7**
3	3	**3**	**4**	**5**	**6**
2	**2**	3	4	5	6
1	**1**	**2**	**3**	**4**	**5**
1	1	2	3	4	5
0	1	2	3	4	5

FIGURE 2. Values of the Hilbert function for D_5

One can check that $h(v_1, v_2)$ for non-negative v_1 and v_2 is (see Figure 2 too):

$$h(v_1,v_2)=\begin{cases} v_1, & \text{if } v_2<3, v_1>0\\ v_1+1, & \text{if } v_2=3, v_1>0\\ v_2-1, & \text{if } v_1<2, v_2\geq 2\\ v_1+v_2-3, & \text{if } v_1\geq 2, v_2\geq 4\\ 0,1,1, & \text{if } v_1=0 \quad \text{and } v_2=0,1,2. \end{cases}$$

2.3. **Some properties of the Hilbert function.** For any $i \in K_0$ let μ_i and δ_i (respectively $\mu(C)$ and $\delta(C)$) be the Milnor number and the delta invariant of C_i (respectively of C) ([1, 10]). Then, cf. [10], $\mu_i = 2\delta_i$, and $\mu(C) + r - 1 = 2\delta(C)$. Define $l = (l_1, \ldots, l_r)$ by

$$l_i = \mu_i + \sum_{j\neq i}(C_j, C_i).$$

It is known that the Alexander polynomial is symmetric in the following sense

$$\Delta(t_1^{-1},\ldots,t_r^{-1}) = \left(\prod t_i^{1-l_i}\right)\cdot\Delta(t_1,\ldots,t_r) \quad \text{for } r>1,$$

$$\Delta(t^{-1}) = t^{-\mu(C)}\Delta(t) \qquad \text{for } r=1.$$

This can be compared with the following.

Lemma 2.3.1. *([12, 6]) The Hilbert function satisfies the following symmetry property:*

$$h(l-v)-h(v)=\delta(C)-|v|, \tag{2.3.2}$$

where $|v|=\sum_{i=1}^r v_i$.

Remark 2.3.3. Consider $v\succeq l$. It follows from (2.3.2) that $h(v)=|v|-\delta(C)$. This can be verified by the identity (2.2.13) as well. Indeed, for $|K|=1$ we have $\sum_{0\le u_i\le v_i-1}\pi^K(u)=v_i-\delta(C_i)$, for $|K|=2$ we have $\sum\pi^{\{i,j\}}(u_i,u_j)=P^{\{i,j\}}(1,1)=(C_i,C_j)$, while $P^K(e_K)=0$ for $|K|>2$. Hence $h(v)=\sum_i(v_i-\delta(C_i))-\sum_{i,j}(C_i,C_j)=|v|-\delta(C)$.

The identity $h(v)=|v|-\delta(C)$ and Lemma 2.2.3 give that $v\in\mathcal{S}$ whenever $v\succeq l$ (hence, in fact, l is the conductor of $\mathcal{S}$). This fact combined again with 2.2.3 gives

Corollary 2.3.4. *For any basic vector e_i and $n\ge l_i$ one has $h(v+(n+1)e_i)-h(v+ne_i)=1$.*

The next property is not used in the present note, nevertheless we add it since it contains the key observation which will be used in a forthcoming article with applications in deformations.

Proposition 2.3.5. *(a) Let f' be a deformation of f, where f and f' are irreducible. Then $h_f(v)\le h_{f'}(v)$ for every v.*

(b) Let a (possibly reducible) curve C' be a deformation of an irreducible curve C. Then $h_{C'}(v)\ge h_C(|v|)$ for every v.

Proof. (a) For any function g the intersection multiplicity of g with f is greater or equal to the intersection multiplicity of g with f'. Therefore $J_{f'}(v)\subset J_f(v)$ and

$$h_f(v)=\operatorname{codim} J_f(v)\le \operatorname{codim} J_{f'}(v)=h_{f'}(v).$$

(b) Consider a function g from $J_{C'}(v)$. Its orders on the components of C' are greater or equal to the corresponding components of v, hence the intersection multiplicity of g with C' is greater or equal to $|v|$. Hence the intersection multiplicity of g with C is greater or equal to $|v|$. Therefore $J_{C'}(v)\subset J_C(|v|)$ and $h_C(|v|)=\operatorname{codim} J_C(|v|)\le \operatorname{codim} J_{C'}(v)=h_{C'}(v)$. $\square$

2.4. **Motivic Poincaré series.** Following [5], we define the motivic Poincaré series of a plane curve singularity $C=\cup_{i=1}^r C_i$ by the formula

$$P_g(t_1,\dots,t_r;q)=\frac{1}{1-q}\sum_{v\in\mathbb{Z}^r}t_1^{v_1}\cdots t_r^{v_r}\sum_{K\subset K_0}(-1)^{|K|}q^{h(v+e_K)}.$$

It follows from the results of [5] that this definition agrees with the integral (1.0.2). In [7], and independently in [12], the following properties are proved: $P_g(t_1,\dots,t_r;q)$ is a rational function with denominator $\prod_{i=1}^r(1-t_iq)$, hence

$$\overline{P}_g(t_1,\dots,t_r;q):=P_g(t_1,\dots,t_r;q)\cdot\prod_{i=1}^r(1-t_iq)$$

is a polynomial. Moreover, $\overline{P}_g$ satisfies the functional equation

$$\overline{P}_g(\frac{1}{qt_1},\dots,\frac{1}{qt_r};q)=q^{-\delta(C)}\prod_i t_i^{-l_i}\cdot\overline{P}_g(t_1,\dots,t_r;q).$$

In [12] this equation was deduced from (2.3.2). Moreover, one can check that for $q=1$ one has $P_g(t_1,\dots,t_r;q=1)=P(t_1,\dots,t_r)$. An explicit algorithm of the computation of $P_g(t_1,\dots,t_r;q)$ in terms of the embedded resolution tree of C is provided in [7].

2.5. **Conclusion.** By the above discussions, the following objects associated with a plane curve singularity carry the same amount of information: Δ, $\mathcal{S}$, H, P and P_g.

3. Central hyperplane arrangements

3.1. Matroids and rank functions.

Definition 3.1.1. (A) ([27]) Let K_0 be a finite set. A function ρ, assigning a non-negative integer to any subset $K \subset K_0$, is called a *rank function*, if

(1) $0 \leq \rho(K) \leq |K|$ (where $|K|$ denotes the cardinality of K).
(2) If $K_1 \subset K_2$ then $\rho(K_1) \leq \rho(K_2)$.
(3) For every pair of subsets K_1 and K_2 one has the following inequality:

$$\rho(K_1 \cap K_2) + \rho(K_1 \cup K_2) \leq \rho(K_1) + \rho(K_2).$$

(B) A *matroid* is a finite set with a rank function on it.
(C) The characteristic polynomial of a matroid $M = (K_0, \rho)$ is defined as

$$\chi_M(t) = \sum_{K \subset K_0} (-1)^{|K|} t^{\rho(K_0) - \rho(K)}.$$

Remark 3.1.2. Some authors define the characteristic polynomial using the Möbius function of a matroid. This definition is equivalent to the present one, see e.g. [27, Theorem 2.4].

Let $h(v)$ denote the Hilbert function of a plane curve singularity. Let us fix $K_0 = \{1, \dots, r\}$ and for every v consider the following function on subsets of K_0:

$$\rho_v(K) := h(v + e_K) - h(v) = \dim J(v)/J(v + e_K).$$

Proposition 3.1.3. *For every v the function ρ_v is a rank function on K_0.*

Proof. Property (1) follows from Lemma 2.2.3. Next, for $K_1 \subset K_2$ one has $\rho_v(K_2) - \rho_v(K_1) = \dim J(v + e_{K_1})/J(v + e_{K_2}) \geq 0$. To prove the third property notice that $J(v + e_{K_1 \cup K_2}) = J(v + e_{K_1}) \cap J(v + e_{K_2})$, and $J(v + e_{K_1}) + J(v + e_{K_2}) \subset J(v + e_{K_1 \cap K_2})$, hence

$$\rho_v(K_1) + \rho_v(K_2) - \rho_v(K_1 \cup K_2) = \dim J(v)/(J(v + e_{K_1}) + J(v + e_{K_2})) \geq \rho_v(K_1 \cap K_2).$$

□

We will call ρ_v the rank function for a the "local matroid M_v". In the space $J(v)$ we have r subspaces $J(v + e_i)$ of codimension 0 or 1. If $v \in \mathcal{S}$, then the set of functions with valuation v can be represented as a complement of a hyperplane arrangement (cf. [12]). If $v \notin \mathcal{S}$, then $J(v) = J(v + e_{i_0})$ for some i_0, cf. Lemma 2.2.3. Moreover, one has the following lemma.

Lemma 3.1.4. *Assume that $J(v) = J(v + e_{i_0})$ for some $i_0 \in K_0$. Then $J(v + e_K) = J(v + e_K + e_{i_0})$ for any $K \subset E$ with $K \not\ni i_0$. Hence*

$$\chi_{M_v}(t) = \sum_{K \subset K_0} (-1)^{|K|} \cdot t^{h(v + e_{K_0}) - h(v + e_K)} = 0.$$

Proof. Use $J(v + e_K + e_{i_0}) = J(v + e_K) \cap J(v + e_{i_0})$ for the first statement, and pairwise cancellation for the second one. □

3.2. **Central hyperplane arrangements and Orlik-Solomon algebras.**

Let us recall some facts about central hyperplane arrangements. Let V be a vector space and let $\mathcal{H} = \{\mathcal{H}_1, \ldots, \mathcal{H}_r\}$ be a collection of linear hyperplanes in V. For a set $K = \{\alpha_1, \ldots, \alpha_k\}$ we define the rank function

$$\rho(K) = \mathrm{codim}(\mathcal{H}_{\alpha_1} \cap \ldots \cap \mathcal{H}_{\alpha_k}).$$

Similarly to Proposition 3.1.3, one can check that ρ is a rank function for a certain matroid. Let us denote by $\chi_{\mathcal{H}}(t)$ its characteristic polynomial.

Lemma 3.2.1. *The class in the Grothendieck ring of varieties of the complement of $\cup_i \mathcal{H}_i$ in V equals*

$$[V \setminus \cup_{i=1}^r \mathcal{H}_i] = \mathbb{L}^{\dim V - \rho(K_0)} \chi_{\mathcal{H}}(\mathbb{L}),$$

where $\mathbb{L}$ denotes the class of the affine line.

Proof. Follows from the inclusion-exclusion formula:

$$[V \setminus \cup_{i=1}^r \mathcal{H}_i] = \sum_{K \subset K_0} (-1)^{|K|} \left[\cap_{\alpha \in K} \mathcal{H}_\alpha\right] = \sum_{K \subset K_0} (-1)^{|K|} \, \mathbb{L}^{\dim V - \rho(K)}. \quad \square$$

To the arrangement $\mathcal{H}$ one can associate the corresponding Orlik-Solomon algebra. Consider the anticommutative algebra $\mathcal{E}$ generated by the variables $z_1, \ldots, z_r$ corresponding to hyperplanes. For any set $K \subset K_0$ we consider the monomial $z_K = z_{\alpha_1} \wedge \ldots \wedge z_{\alpha_k}$, where $K = \{\alpha_1, \ldots, \alpha_k\}$. We can equip $\mathcal{E}$ with the natural differential ∂ sending z_i to 1, namely

$$\partial(z_K) = \sum_{i=1}^{k} (-1)^{i-1} z_{K \setminus \{\alpha_i\}}.$$

Definition 3.2.2. We call the set K *dependent*, if the equations of the corresponding hyperplanes are linearly dependent. Otherwise K is called *independent*.

The Orlik-Solomon ideal $\mathcal{I}$ is the ideal in $\mathcal{E}$ generated by the elements ∂z_K for all dependent sets K. The Orlik-Solomon algebra is the quotient $\mathcal{A} = \mathcal{E}/\mathcal{I}$.

Theorem 3.2.3. *(*[18, Theorem 5.2]*) The integral cohomology ring of the complement $V \setminus \cup_{i=1}^r \mathcal{H}_i$ is isomorphic to the Orlik-Solomon algebra $\mathcal{E}/\mathcal{I}$. It has no torsion, and its Poincaré polynomial is given by the formula*

$$P(\mathcal{H}, t) = (-t)^{r(E)} \cdot \chi_{\mathcal{H}}(-t^{-1}) = \sum_{K \subset K_0} (-1)^{|K|} (-t)^{\rho(K)}.$$

As a corollary, we conclude that the homology of $V \setminus \cup_{i=1}^r \mathcal{H}_i$ is defined by its class in the Grothendieck ring. The same is true for its projectivization (see below).

We will need the following deformations of the differential on $\mathcal{E}$.

Definition 3.2.4. Let us define the following operator:

$$\partial_U : \mathcal{E}[U] \to \mathcal{E}[U], \quad \partial_U(z_K) = \sum_{i=1}^{k} (-1)^{i-1} U^{\rho(K) - \rho(K \setminus \{\alpha_i\})} z_{K \setminus \alpha_i},$$

where U is a formal parameter and $K = \{\alpha_1, \ldots, \alpha_k\}$.

Note that $\rho(K) - \rho(K \setminus \{\alpha_i\}) \in \{0, 1\}$, hence ∂_U decomposes into a sum of two components

$$\partial_U = \partial_0 + U \partial_1, \quad \text{with} \quad \partial_0 + \partial_1 = \partial. \tag{3.2.5}$$

Lemma 3.2.6. *The operator ∂_U is a differential on $\mathcal{E}[U]$, that is, $\partial_U^2 = 0$. In particular, the following identities hold:*
$$\partial_0^2 = \partial_1^2 = 0, \quad \partial_0\partial_1 + \partial_1\partial_0 = 0.$$

Proof. Straightforward. □

Let $\mathcal{J}$ and $\mathcal{J}^\perp$ denote the *subspaces* of $\mathcal{E}$ spanned by the elements z_K for all dependent, respectively independent subsets K. Clearly $\mathcal{E} = \mathcal{J} \oplus \mathcal{J}^\perp$.

Lemma 3.2.7. *The following statements hold:*

(a) *(*[18], *Lemma 2.7)* $\mathcal{I} = \mathcal{J} + \partial\mathcal{J}$.
(b) $\partial_0\mathcal{J}^\perp = 0$, *hence* $\operatorname{Im}\partial_0 = \partial_0\mathcal{J}$.
(c) $\partial_1\mathcal{J} \subset \mathcal{J}$, *hence* $\mathcal{I} = \mathcal{J} + \partial\mathcal{J} = \mathcal{J} + \partial_0\mathcal{J}$.
(d) $\operatorname{Ker}\partial_0 = \mathcal{J}^\perp + \operatorname{Im}\partial_0$.
(e) *There exist subspaces* $A \subset \mathcal{J}, B \subset \mathcal{J}^\perp$ *such that* $\operatorname{Im}\partial_0 = A \oplus B$.

Proof. The claims (b) and (c) are clear. Let us prove (d). The inclusion $\mathcal{J}^\perp + \operatorname{Im}\partial_0 \subset \operatorname{Ker}\partial_0$ is also clear, hence we need to prove that if $\partial_0(\phi) = 0$ then there exists $\widetilde{\phi} \in \mathcal{J}^\perp$ such that $\phi - \widetilde{\phi} \in \operatorname{Im}(\partial_0)$.

Let us call z_i *essential* in a monomial $z_i \wedge z_K$, if $\rho(\{i\} \sqcup K) = \rho(K) + 1$, and *redundant* otherwise. Let us decompose $\phi = z_1\wedge\phi_1 + z_1\wedge\phi_2 + \phi_3$, where z_1 is essential in every monomial of $z_1 \wedge \phi_1$, redundant in every monomial of $z_1 \wedge \phi_2$, and ϕ_3 contains no z_1.

Then
$$\partial_0(\phi) = z_1 \wedge \psi + \phi_2 + \partial_0(\phi_3)$$
for some ψ, and neither ϕ_2 nor $\partial_0(\phi_3)$ contain z_1. Hence, if $\partial_0(\phi) = 0$ then $\phi_2 = -\partial_0(\phi_3)$.

Since z_1 is redundant in every monomial in $z_1 \wedge \partial_0(\phi_3)$, it is redundant in every monomial in $z_1 \wedge \phi_3$. Therefore
$$\partial_0(z_1 \wedge \phi_3) = \phi_3 - z_1 \wedge \partial_0(\phi_3) + z_1 \wedge \eta,$$
where z_1 is essential in every monomial of $z_1 \wedge \eta$. Indeed, if $\alpha_j \in K$, z_{α_j} is redundant in $K \cup \{1\}$ and essential in K, then z_1 is essential in $K \cup \{1\} \setminus \{\alpha_j\}$.

We conclude that
$$\phi - \partial_0(z_1 \wedge \phi_3) = z_1 \wedge (\phi_1 - \eta)$$
and z_1 is essential in every monomial in the right hand side. Now $\partial_0(\phi) = -z_1\wedge\partial_0(\phi_1-\eta) = 0$, hence $\partial_0(\phi_1 - \eta) = 0$. Then we can repeat the procedure inductively replacing ϕ by $\phi_1 - \eta$, and z_1 by z_2, etc. At the end we reduce ϕ modulo $\operatorname{Im}(\partial_0)$ to an element of $\mathcal{E}$ where all z_i are essential; such an element belongs to $\mathcal{J}^\perp$.

Next, we prove (e). Recall that $\operatorname{Im}\partial_0 = \partial_0 J$ and K is dependent iff $\rho(K) < |K|$. If the monomial $z_{K'}$ appears in $\partial_0(z_K)$ then $\rho(K) = \rho(K')$ and $|K'| = |K| - 1$. Therefore $\partial_0(z_K) \in \mathcal{J}^\perp$ if $\rho(K) = |K| - 1$, and $\partial_0(z_K) \in \mathcal{J}$ otherwise. □

We will need the following two results in the construction of lattice homology.

Theorem 3.2.8. *The Orlik-Solomon algebra is isomorphic to the homology of the differential ∂_0:*
$$H_*(\mathcal{E}, \partial_0) = \mathcal{E}/\mathcal{I} \simeq H^*\left(V \setminus \cup_{i=1}^r \mathcal{H}_i\right).$$

Proof. By Lemma 3.2.7 one has the isomorphisms
$$\operatorname{Ker}\partial_0 = \mathcal{J}^\perp + \operatorname{Im}\partial_0, \quad \operatorname{Im}\partial_0 = \partial_0\mathcal{J},$$
hence
$$H_*(\mathcal{E}, \partial_0) = (\mathcal{J}^\perp + \partial_0\mathcal{J})/\partial_0\mathcal{J} \simeq \mathcal{J}^\perp/(\partial_0\mathcal{J} \cap \mathcal{J}^\perp) \simeq \mathcal{E}/(\mathcal{J} + \partial_0\mathcal{J}).$$

The last identity follows from the splitting in Lemma 3.2.7(e). Then use 3.2.7(c). □

Remark 3.2.9. The space $\mathcal{E}$ is bigraded: the first grading assigns $|K|$ to z_K, and the second grading assigns $\rho(K)$ to z_K. It is clear that ∂_0 decreases the first grading by 1 and preserves the second grading, hence its homology are naturally bigraded.

On the other hand, it follows from the proof of Theorem 3.2.8 that $H^*(\mathcal{E}, \partial_0)$ can be identified with a quotient of $\mathcal{J}^{\perp}$. Since $\mathcal{J}^{\perp}$ is spanned by the independent monomials, the gradings induced by $|K|$ and $\rho(K)$ coincide on $H^*(\mathcal{E}, \partial_0)$.

Let us describe the homological gradings on $\mathcal{E}[U]$ compatible with ∂_U. One can check that the equation

$$\deg_\lambda(U^m z_K) = |K| + \lambda(m + \rho(K)) \tag{3.2.10}$$

defines a unique grading on $\mathcal{E}[U]$ such that ∂_U has degree (-1) and the operator of multiplication by U has degree λ.

Let $P(\mathcal{H}, t)$ and $P(\mathbb{P}\mathcal{H}, t)$ denote the Poincaré polynomials of the linear arrangement $V \setminus \cup_{i=1}^r \mathcal{H}_i$ and of the projective arrangement $\mathbb{P}V \setminus \cup_{i=1}^r \mathbb{P}\mathcal{H}_i$.

Theorem 3.2.11. *Let $\partial_1^{\mathcal{A}}$ be the operator induced by ∂_1 on $H_*(\mathcal{E}, \partial_0)$. Then $H_*(\mathcal{E}[U], \partial_U) \simeq \operatorname{Ker} \partial_1^{\mathcal{A}} \subset H_*(\mathcal{E}, \partial_0)$. Furthermore, the Poincaré polynomial of $H_*(\mathcal{E}[U], \partial_U)$ in grading $\deg_\lambda$ is equal to*

$$P(\mathcal{E}[U], \partial_U, t) = P(\mathbb{P}\mathcal{H}, t^{\lambda+1}). \tag{3.2.12}$$

Proof. The following equation holds, cf. [19, Corollary 3.58]:

$$P(\mathbb{P}\mathcal{H}, t) = \frac{P(\mathcal{H}, t)}{1+t}. \tag{3.2.13}$$

From [19, Lemma 3.42] one gets that $\partial_1^{\mathcal{A}}$ is acyclic on $\mathcal{A} := H_*(\mathcal{E}, \partial_0)$. Since by Theorem 3.2.8 the Poincaré polynomial (in the grading induced by $|K|$) of $\mathcal{A}$ equals to $P(\mathcal{H}, t)$, and the operator $\partial_1^{\mathcal{A}}$ has degree -1, the Poincaré polynomial of $\operatorname{Ker} \partial_1^{\mathcal{A}}$ equals

$$P_{\mathrm{Ker}} = \frac{P(\mathcal{H}, t)}{1+t}. \tag{3.2.14}$$

Since $\partial_U = \partial_0 + U\partial_1$, there exists a spectral sequence starting from $H_*(\mathcal{E}[U], \partial_0) = \mathcal{A}[U]$ and converging to $H_*(\mathcal{E}[U], \partial_U)$. It actually converges at E_2 page, which is equal to

$$H_*(\mathcal{E}[U], \partial_U) \simeq H_*(\mathcal{A}[U], U\partial_1) \simeq \operatorname{Ker} \partial_1^{\mathcal{A}}.$$

It follows from the Remark 3.2.9 that $\deg_\lambda(z_K) = (\lambda+1)|K|$ on $\mathcal{A}$. The theorem now follows from the comparison of (3.2.13) and (3.2.14). □

Example 3.2.15. Consider an arrangement of r generic lines through the origin in $V = \mathbb{R}^2$. The differential ∂_U has a form

$$\partial_U(1) = 0,\ \partial_U(z_i) = U,\ \partial_U(z_i \wedge z_j) = U(z_i - z_j),$$

$$\partial_U(z_K) = \partial(z_K) \quad \text{for} \quad |K| \geq 3.$$

The homology of ∂_U is spanned by $1, z_1 - z_2, \ldots, z_1 - z_r$. On the other hand, $\mathbb{P}V \setminus \mathbb{P}\mathcal{H}$ is the complement to r points in $\mathbb{CP}^1$, homotopically equivalent to the bouquet of $(r-1)$ circles.

4. Lattice homology

Lattice cohomology associated with the intersection lattice of a resolution of a normal surface singularity was introduced in [17], as a topological invariant of negative definite plumbed 3–manifolds. For a possible generalization to algebraic knots, see the recent manuscript [25]. In this section we introduce another homology theory associated with curve singularities, where the lattice and the corresponding weight function has a different nature. In order to make a distinction between the two cases we will call the present theory *lattice cohomology of curve singularities via normalization*.

In fact, the definitions below extend identically to any, not necessarily plane curve singularity, that is, even for a germ $(C,0)$ which does not have any local *embedded link* in the 3–sphere.

Let us fix a local plane curve singularity $(C,0)$, where $C = C_1 \cup \cdots \cup C_r$. Recall that $v \mapsto h(v)$ denotes its Hilbert function for $v \in \mathbb{Z}^r$.

4.1. **The lattice complex.** We will use the cubes of the lattice $\mathbb{Z}^r$. Every such cube $\square(v,K)$, $v \in \mathbb{Z}^r, K \subset K_0$ is defined as

$$\square = \square(v,K) = \{x \in \mathbb{R}^r : v \preceq x \preceq v + e_K\},\ \dim \square(v,K) = |K|.$$

Definition 4.1.1. We define the weight function of a cube by the formula

$$h(\square) = \max\{h(x) : x \in \square \cap \mathbb{Z}^r\}.$$

Since $h(v)$ is non-decreasing, in fact, one has $h(\square(v,K)) = h(v+e_K)$.

Let us denote by Q_q the set of all q-dimensional cubes (with their natural orientation). The oriented boundary of a cube (as in the classical cubical homology), can be written as

$$\partial \square(v,K) = \sum_{i \in K} \varepsilon_i \cdot \square(v, K \setminus \{i\}),\ \varepsilon_i = \pm 1.$$

We will abbreviate this equation as $\partial(\square) = \sum_i \varepsilon_i \square_i$.

Definition 4.1.2. The lattice complex $\mathcal{L}^-$ is a free $\mathbb{Z}[U]$-module generated by all $\square = \square(v,K)$ with the following $\mathbb{Z}[U]$–linear differential:

$$\partial_U(\square) = \sum_i \varepsilon_i U^{h(\square)-h(\square_i)} \square_i. \tag{4.1.3}$$

One can verify that $\partial_U^2 = 0$ (compare with lemma 3.2.6).

We set $\deg U = -2$ and we introduce the homological grading of a generator by

$$\deg(U^k \square) = -2k + \dim(\square) - 2h(\square).$$

Then ∂_U decreases the homological grading by 1.

4.2. **Filtration.**

Definition 4.2.1. We define the r-index filtration on the complex $\mathcal{L}^-$ as follows: the subcomplex $\mathcal{L}^-(u) = \mathcal{L}^-(u_1, \ldots, u_r)$ is generated over $\mathbb{Z}[U]$ by all the cubes $\square(v,K)$ with $v \succeq u$. (It is easy to see that ∂_U preserves the filtration, so $\mathcal{L}^-(u)$ is a subcomplex of $\mathcal{L}^-$ for all u.)

Theorem 4.2.2.

(a) The homology of $\mathcal{L}^-(u)$ is isomorphic to $\mathbb{Z}[U]$. It is generated by the 0-dimensional cube at the lattice point u, which has homological degree $-2h(u)$.

(b) The inclusion $\mathcal{L}^-(0) \subset \mathcal{L}^-$ induces an isomorphism at the level of homology. In particular, the homology of $\mathcal{L}^-$ is $\mathbb{Z}[U]$.

Proof. (a) For every $k \geq h(u)$ let us define the topological space $S_k(u) := \cup \square(v, K)$, where the union is over those cubes from $\mathcal{L}^-(u)$ (hence satisfying $v \succeq u$) with $h(\square(v, K)) = h(v + e_K) \leq k$. (Here we think about $\square(v, K)$ as a cube in $\mathbb{R}^r$.) By Corollary 2.3.4, $S_k(u)$ is compact.

Similarly as in [17, Theorem 3.1.12], we show the following isomorphism of $\mathbb{Z}$–modules for any $q \in \mathbb{Z}$:

$$H_q(\mathcal{L}^-(u)) = \bigoplus_{\substack{k \geq h(u) \\ q'-2k=q}} H_{q'}(S_k(u), \mathbb{Z}). \tag{4.2.3}$$

This can be proved as follows. Let $\mathcal{C}_*(S_k(u))$ be the usual cubical chain complex of $S_k(u)$. We define the $\mathbb{Z}$–linear morphism $\Phi : \mathcal{L}^-(u) \to \oplus_{k \geq h(u)} \mathcal{C}_*(S_k(u))$ by $U^l \square(v, K) \mapsto \square(v, K)$, where the last cube $\square(v, K)$ is considered in $\mathcal{C}_{|K|}(S_k(u))$ with $k = l + h(v + e_K)$. This is a linear isomorphism. Indeed, any cube $\square(v, K) \in \mathcal{C}_{|K|}(S_k(u))$ (where $k \geq h(v + e_K)$) can be represented as $\Phi(U^{k-h(v+e_K)} \square(v, K))$. Moreover, if $\Phi(U^l \square(v, K)) \in \mathcal{C}_{|K|}(S_k(u))$, then $\Phi(\partial_U(U^l \square(v, K))) = \partial \Phi(U^l \square(v, K)) \in \mathcal{C}_{|K|-1}(S_k(u))$ (where ∂ means the usual boundary operator of $\mathcal{C}_*$).

Furthermore, multiplication by U in $\mathcal{L}^-(u)$ corresponds to the operator induced by the inclusion $S_k \hookrightarrow S_{k+1}$ at the level of $\oplus_{k \geq h(u)} \mathcal{C}_*(S_k(u))$.

Hence, Φ induces a morphism at the level of homology. If the homological degree $-2l + |K| - 2h(v + e_K)$ of $U^l \square(v + e_K)$ is denoted by q, then its homological class is sent by Φ into $H_{q'}(S_k)$, where $q' = |K|$ and $2k = 2(l + h(v + e_K)) = |K| - q = q' - q$. Hence (4.2.3) follows.

Next, we prove that $S_k(u)$ is contractible for all k. Indeed, $h(v)$ is non–decreasing, hence if $\square(v, K)$ is in $S_k(u)$, then the set $S_k(u)$ contains the whole parallelepiped $\{x : u \preceq x \preceq v + e_K\}$. Such a space can be contracted to the lattice point u.

In particular, in (4.2.3) q' should be zero, $k = -2k$ and $k \geq h(u)$, and $H_0(S_k) = \mathbb{Z}$. This means that $H_q(\mathcal{L}^-(u))$ is zero unless $q = -2h(v) - 2l$ for $l \geq 0$, and in this case it it $\mathbb{Z}$ corresponding to the generator $\square(u, \emptyset)$ considered in $S_{h(u)+l}$; or, in the homology of $\mathcal{L}^-(u)$, to the class of $U^l \square(u, \emptyset)$. Hence

$$H_*(\mathcal{L}^-(u)) = \mathbb{Z}[U] \cdot \square(u, \emptyset).$$

(b) For $1 \leq p \leq r$ define the sub-complex $\mathcal{L}_p^-$ of $\mathcal{L}^-$ generated over $\mathbb{Z}[U]$ by cubes $\square(v, K)$ with $v = (v_1, \ldots, v_r)$, $v_i \geq 0$ for $1 \leq i \leq p$. Then $\mathcal{L}_r^- = \mathcal{L}^-(0)$ and we also set $\mathcal{L}_0^- := \mathcal{L}^-$. One checks that $\mathcal{L}_p^- \subset \mathcal{L}_{p-1}^-$ is a homotopy equivalence, hence (b) follows by induction on p. $\square$

Remark 4.2.4. A similar statement for the Heegaard-Floer homology HF^- has been obtained in [9] for links such that the sufficiently large surgery of S^3 along its components is an L-space. Such a property is known for algebraic knots, see [8] and Proposition 4.4.3 below. It is not known if this property holds for general algebraic links.

4.3. **Lattice homology (via normalization).**

Definition 4.3.1. We define the multi-graded complex $\operatorname{gr} \mathcal{L}^- = \oplus_v \operatorname{gr}_v \mathcal{L}^-$, where

$$\operatorname{gr}_v \mathcal{L}^- = \mathcal{L}^-(v) / \textstyle\bigoplus_{i=1}^r \mathcal{L}^-(v + e_i).$$

and the homology groups $HL^- = \oplus_v HL^-(v)$, where

$$HL^-(v) = H_*(\operatorname{gr}_v \mathcal{L}^-).$$

Theorem 3.2.3 and equation (3.2.12) imply the following

Theorem 4.3.2. *Consider the motivic Poincaré series of* C

$$P_g(t_1,\dots,t_r;q)=\sum_v H_v(q)t^v, \ \ \textit{where} \ \ H_v(q)=\sum_{K\subset K_0}(-1)^{|K|}\frac{q^{h(v+e_K)}-q^{h(v)}}{1-q}.$$

Then the Poincaré series of $\mathrm{gr}_v\,\mathcal{L}^-$ *satisfies*

$$P_v^{\mathcal{L}^-}(t):=\sum_i t^i\dim H_i(\mathrm{gr}_v\,\mathcal{L}^-)=(-t)^{-h(v)}H_v(-t^{-1}). \tag{4.3.3}$$

Proof. Assume $v\notin\mathcal{S}$ and set $i_0\in K_0$ as in Lemma 3.1.4. Let $\phi:\mathrm{gr}_v\,\mathcal{L}^-\to\mathrm{gr}_v\,\mathcal{L}^-$ be defined by

$$\phi(\square(v,K))=\begin{cases}\square(v,K\cup i_0) & \text{if } i_0\notin K\subset K_0,\\ 0 & \text{if } i_0\in K\subset K_0.\end{cases}$$

Then ϕ realizes a homotopy between the identity and the zero map, that is, $\partial_U\,\phi+\phi\,\partial_U=id$ on $\mathrm{gr}_v\,\mathcal{L}^-$. Hence $H_*(\mathrm{gr}_v\,\mathcal{L}^-)=0$. On the other hand, by Lemma 3.1.4, $H_v(q)=0$ too.

Next, assume that $v\in\mathcal{S}$. Then $J(v+e_i)\subset J(v)$ is a hyperplane for any i. Let $(\mathcal{E}[U],\partial_U)$ be the complex associated with this hyperplane arrangement $\mathcal{H}_v$, as in subsection 3.2. By (3.2.3) and (3.2.13) we have

$$P(\mathcal{H}_v,t)=\sum_{K\subset K_0}(-1)^{|K|}(-t)^{\rho_v(K)},\ P(\mathbb{P}\mathcal{H}_v,t)=\frac{1}{1+t}\sum_{K\subset K_0}(-1)^{|K|}(-t)^{\rho_v(K)}.$$

Let $\overline{\partial}_U:\mathrm{gr}_v\,\mathcal{L}^-\to\mathrm{gr}_v\,\mathcal{L}^-$ be the boundary operator induced by $\partial_U:\mathcal{L}^-\to\mathcal{L}^-$. Then one has an isomorphism $\psi:\mathrm{gr}_v\,\mathcal{L}^-\to\mathcal{E}[U]$, $\psi(\square(v,K))=z_K$, such that $\partial_U\,\psi=\psi\,\overline{\partial}_U$. Hence $H_*(\mathrm{gr}_v\,\mathcal{L}^-)=H_*(\mathcal{E}[U])$. Since U has homological degree (-2), by (3.2.10) the gradings on $\mathrm{gr}_v\,\mathcal{L}^-$ and $\mathcal{E}[U]$ are defined by the equations

$$\deg(U^me_K)=|K|-2(m+\rho_v(K)),\deg(U^m\square(v,K))=|K|-2(m+h(v+e_K))).$$

Since $\rho_v(K)=h(v+e_K)-h(v)$, the isomorphism ψ has homological degree $-2h(v)$.

Hence, at Poincaré polynomial level by (3.2.12) one gets the following equations:

$$P_v^{\mathcal{L}^-}(t)=t^{-2h(v)}P(\mathcal{E}[U],\partial_U,t)=t^{-2h(v)}P(\mathbb{P}\mathcal{H}_v,t^{-1})=\frac{t^{-2h(v)}}{1+t^{-1}}P(\mathcal{H}_v,t^{-1})=$$

$$\frac{t^{-2h(v)}}{1+t^{-1}}\cdot\sum_{K\subset K_0}(-1)^{|K|}(-t)^{-\rho_v(K)}=\frac{(-t)^{-h(v)}}{1+t^{-1}}\cdot\sum_{K\subset K_0}(-1)^{|K|}(-t)^{-h(v+e_K)}.\quad\square$$

Corollary 4.3.4. *A point* v *belongs to the semigroup* $\mathcal{S}$ *if and only if* $HL^-(v)\neq 0$.

Proof. If $v\notin\mathcal{S}$, the statement follows from the Lemma 3.1.4. Suppose that $v\in S$, then $h(v+e_i)=h(v)+1$ for all i and $h(v+e_K)\geq h(v)+1$ for all subsets K. Therefore $H_v(q)=q^{h(v)}-\ldots$, hence

$$P_v^{\mathcal{L}^-}(t)=(-t)^{-h(v)}H_v(-t^{-1})=t^{-2h(v)}+\dots$$

One can also check that the class of the point $a_v=\square(v,\emptyset)$ does not vanish in $HL^-(v)$. $\square$

Remark 4.3.5. Consider the filtration $\{F_n\}_{n\in\mathbb{Z}}$ with sub-complexes F_n of $\mathcal{L}^-$, where F_n is generated over $\mathbb{Z}[U]$ by cubes $\square(v,K)$ with $|v|\geq n$. Then $\oplus_nF_n/F_{n+1}$ has a natural double grading, and in fact, it is isomorphic with $\mathrm{gr}\,\mathcal{L}^-$. This shows that there exists a spectral sequence

$$E^1=HL^-=H_*(\mathrm{gr}\,\mathcal{L}^-)\ \Rightarrow\ E^\infty=H_*(\mathcal{L}^-)=\mathbb{Z}[U].$$

4.4. **Example. One-component case.**

Suppose that C has only one component. Let us compute $HL^-(v)$ from its Hilbert function $h(v)$. To simplify notations, we will abbreviate $\square(v,\emptyset)=a_v$, $\square(v,\{1\})=\alpha_v$. We have two different cases.

a) If $h(v)=h(v+1)$ then $\partial_U(\alpha_v)=a_v$, hence $HL^-(v)=0$.
b) If $h(v)=h(v+1)-1$ then $\partial_U(\alpha_v)=Ua_v$, hence $HL^-(v)=\mathbb{Z}\langle a_v\rangle$. This generator has homological degree $-2h(v)$.

This means the following facts.

(4.4.1)
$$HL^-(v)=\begin{cases} 0 & \text{if } v\notin\mathcal{S}\\ \mathbb{Z}\langle a_v\rangle & \text{if } v\in\mathcal{S}\quad \text{(of homological degree } -2h(v)).\end{cases}$$

Hence $E^1=E^\infty=\mathbb{Z}[U]$ (as a $\mathbb{Z}$-module).

It is interesting to consider the complex $\mathcal{L}^-_{U=0}$ too (obtained from $\mathcal{L}^-$ via substitution $U=0$), generated over $\mathbb{Z}$ by the cubes and boundary operator given by (4.1.3) with substitution $U=0$. Then $H_*(\mathcal{L}^-_{U=0})=\mathbb{Z}$ (generated by the zero dimensional cube a_0). Moreover, the filtration $F'_n:=F_n|_{U=0}$ induces a spectral sequence, where F'_n/F'_{n+1} is generated over $\mathbb{Z}$ by all a_v and α_v, and the only non-trivial components of the boundary map are the isomorphisms $\mathbb{Z}\langle\alpha_v\rangle\to\mathbb{Z}\langle a_v\rangle$ for any $v\notin\mathcal{S}$. In particular, the E^1 is $\oplus_{v\in\mathcal{S}}\mathbb{Z}\langle a_v,\alpha_v\rangle$. The non-trivial components of the $d^1: E^1\to E^1$ operator are the isomorphisms $\mathbb{Z}\langle\alpha_v\rangle\to\mathbb{Z}\langle a_{v+1}\rangle$ whenever both v and $v+1$ are elements of $\mathcal{S}$. Hence the E^2 term of the spectral sequence

(4.4.2)
$$E^2(v)=\begin{cases} \mathbb{Z}\langle a_v\rangle & \text{if } v\in\mathcal{S} \text{ and } v-1\notin\mathcal{S} \quad \text{(of homological degree } -2h(v))\\ \mathbb{Z}\langle \alpha_v\rangle & \text{if } v\in\mathcal{S} \text{ and } v+1\notin\mathcal{S} \quad \text{(of homological degree } 1-2h(v))\\ 0 & \text{otherwise.}\end{cases}$$

The parity of the homological degree provides a $\mathbb{Z}_2$ grading $\{E^2_\epsilon\}_{\epsilon\in\{0,1\}}$ of E^2 (where ϵ has the same parity as the homological degree). Then

$$\sum_{v,\epsilon}(-1)^\epsilon\,\mathrm{rank}(E^2_\epsilon(v))\,t^{v+\epsilon} \;=\; \Delta_C(t).$$

The E^∞ term is $H_*(\mathcal{L}^-_{U=0})=\mathbb{Z}$.

The U–action on HL^- is $Ua_v=a_{v+1}$ if $v+1\in\mathcal{S}$, and $Ua_v=0$ otherwise. This description has the following consequences.

(a) E^2 is supported in $[0,\mu]$, where μ is the Milnor number of C.
(b) $v\mapsto\mu-v$ is a symmetry of E^2_ϵ which preserves the ϵ–degree.
(c) From both HL^- and E^2 one can recover the semigroup $\mathcal{S}$ of C.
(d) $\mathrm{Ker}(U)[v\mapsto v+1]=E^2_1(v)$ and $\mathrm{Coker}(U)[v\mapsto v+1]=E^2_0(v+1)$, hence

$$0\to E^2_1(v)[-1]\to HL^-(v)\xrightarrow{U} HL^-(v+1)\to E^2_0(v+1)\to 0.$$

The above facts are compatible with [21, 7] regarding the Heegaard Floer homology. Moreover, we have the following result.

Proposition 4.4.3. *For a locally irreducible plane curve singularity the homology HL^- is isomorphic as $\mathbb{Z}[U]$-module to the Heegaard–Floer homology HF^- of its link.*

Proof. In follows from the results of [21] that the homology HF^- of an *L-space knot* are completely determined as $\mathbb{Z}[U]$-module by its Alexander polynomial. It is proven in [8] that all

$$\begin{tikzcd}[column sep=small] h \arrow[r, dash, dashed] \arrow[d, dash, dashed] & h \arrow[d, dash, dashed] \\ h \arrow[r, dash, dashed] & h \end{tikzcd} \quad \begin{tikzcd}[column sep=small] (h+1) \arrow[r, dash, dashed] \arrow[d, dash, dashed] & (h+1) \arrow[d, dash, dashed] \\ h \arrow[r, dash, dashed] & h \end{tikzcd} \quad \begin{tikzcd}[column sep=small] h \arrow[r, dash, dashed] \arrow[d, dash, dashed] & (h+1) \arrow[d, dash, dashed] \\ h \arrow[r, dash, dashed] & (h+1) \end{tikzcd}$$

(a) (b) (c)

$$\begin{tikzcd}[column sep=small] (h+1) \arrow[r, dash, dashed] \arrow[d, dash, dashed] & (h+1) \arrow[d, dash, dashed] \\ h \arrow[r, dash, dashed] & (h+1) \end{tikzcd} \quad \begin{tikzcd}[column sep=small] (h+1) \arrow[r, dash, dashed] \arrow[d, dash, dashed] & (h+2) \arrow[d, dash, dashed] \\ h \arrow[r, dash, dashed] & (h+1) \end{tikzcd}$$

(d) (e)

FIGURE 3. Possible local behaviours of the Hilbert function

algebraic knots belong to the class of L-space knots, hence one can use the algorithm from [21] to compute their Heegaard-Floer homology. Finally, one can show similarly to [7, Theorem 6] that the algorithm from [21] agrees with the equation (4.4.1), since the semigroup $\mathcal{S}$ can be reconstructed from the Alexander polynomial. $\square$

4.5. **Example. Two-component case.** Let us compute the homology HL^- for a two-component link. To simplify notations, we will abbreviate $\square(v,\emptyset) = a_v$, $\square(v,\{1\}) = \alpha_v$ $\square(v,\{2\}) = \beta_v$. $\square(v,\{1,2\}) = \Gamma_v$. We have five different cases (see Figure 3).

a) $h(v) = h(v+e_1) = h(v+e_2) = h(v+e_1+e_2)$. In this case we have

$$\partial_U(\alpha_v) = \partial_U(\beta_v) = a_v,\ \partial_U(\Gamma_v) = \alpha_v - \beta_v,$$

hence $HL^-(v) = 0$.

b) $h(v) = h(v+e_1) = h(v+e_2) - 1 = h(v+e_1+e_2) - 1$. In this case we have

$$\partial_U(\alpha_v) = a_v, \partial_U(\beta_v) = Ua_v,\ \partial_U(\Gamma_v) = U\alpha_v - \beta_v,$$

hence $HL^-(v) = 0$.

c) $h(v) = h(v+e_1) - 1 = h(v+e_2) = h(v+e_1+e_2) - 1$. Analogously to (b), $HL^-(v) = 0$.

d) $h(v) = h(v+e_1) - 1 = h(v+e_2) - 1 = h(v+e_1+e_2) - 1$. In this case we have

$$\partial_U(\alpha_v) = \partial_U(\beta_v) = Ua_v,\ \partial_U(\Gamma_v) = \alpha_v - \beta_v,$$

hence $HL^-(v) = \mathbb{Z}\langle a_v\rangle$ of homological degree $-2h(v)$.

e) $h(v) = h(v+e_1) - 1 = h(v+e_2) - 1 = h(v+e_1+e_2) - 2$. In this case we have

$$\partial_U(\alpha_v) = \partial_U(\beta_v) = Ua_v,\ \partial_U(\Gamma_v) = U\alpha_v - U\beta_v,$$

hence $HL^-(v) = \mathbb{Z}\langle a_v, \alpha_v - \beta_v\rangle$ of homological degrees $-2h(v), -1-2h(v)$.

In the cases (a)-(c) the point v does not belong to the semigroup $\mathcal{S}$, so $HL^- = 0$. In case (e) the Euler characteristic of $HL^-(v)$ (and the corresponding coefficient in the Alexander polynomial) vanishes, but the homology and the coefficient in the motivic Poincaré series do not vanish. This case appears, for example, for all v in the conductor of C.

Using this computation and Figures 1 and 2, one can compute the HL^- for the singularities of types A_3 and D_5. The analogous computation for the two-component singularity A_{2n-1} agrees with the computations of the Heegaard-Floer homology in [23].

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